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Benchmarking of the Split-Step Fourier Method on Solving a Soliton Propagation Equation in a Nonlinear Optical Medium

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ABSTRACT

Benchmarking of the numerical split-step Fourier method in solving a soliton propagation equation in a nonlinear optical medium is considered. This study is carried out by comparing the solutions calculated by numerics with those obtained by analytics. In particular, the soliton propagation equation used as the object of observation is the nonlinear Schrödinger (NLS) equation, which describes optical solitons in optical fiber. By using the split-step Fourier method, we show that the split-step Fourier method is accurate. We also confirm that the nonlinear and dispersion parameters of the optical fiber influence the soliton propagation.

1. INTRODUCTION

Solitons are solitary waves, often referred as nonlinear waves, which are envelopes or single wave packet or pulses that can maintain their shape stability when propagating at a constant velocity in a medium (Agrawal, 2013; Song et al., 2019). Based on the historical perspective reviewed by Kasman (2018), the term “soliton” was used initially to refer to the phenomenon of a stable isolated water wave, spreading in an unchanging shape along a narrow channel or canal. In the last few decades, solitons were investigated and developed in the field of optics by the great work of Chiao et al. (1964) and Zakharov and Shabat (1972). They discovered a unique solution (soliton) of wave propagation equations in nonlinear optical media (Kajzar and Reinisch, 2006).

The discovery of solitons in the nonlinear optical medium is known to have led to a new field of applied research. Solitons are now part of natural phenomena that are investigated in various scientific disciplines. In the optical field, soliton has become a new trend of scientific research in the modern era. Soliton in this field was initially confirmed by Hasegawa in 1973 in optical fiber as an alternative to overcome the loss constraints that often occur when the optical fiber used as a transmission medium (Agrawal, 2001). In addition to optical fibers, soliton also found in other optical fields, such as lenses (Grahelj, 2010) and photonic crystals (Kisvhar and Agrawal, 2003; Arteaga-Sierra et al., 2018).

Research progress on solitons in the optical field continues rapidly, which is indicated by a variety of topics that have been widely explored (Agrawal, 2013). Based on the various reviews, solitons in the optical medium are known as optical solitons (Kisvhar and Agrawal, 2003; Kajzar and

Reinisch, 2006; Grahelj, 2010; Agrawal, 2013; Song et al., 2019; Wazwaz and Kaur, 2019). Theoretically, it is described by a nonlinear wave propagation equation, .i.e the nonlinear Schrödinger (NLS) equation (Agrawal, 2013; Ripai et al., 2019; Wazwaz and Kaur, 2019). Because of its role in optical technology development, such as optical fibers, the NLS equation is essential to study because it can explain the dynamics of the optical solitons.

The NLS equation is known to be a mathematical model of optical solitons in the form of nonlinear partial differential (Grahelj, 2010; Agrawal, 2013; Wazwaz and Kaur, 2019). This equation is so-called because it has similarities with the one that appeared in the study of quantum physics. However, the potential form in the considered equation is nonlinear (Agrawal, 2013). Thus, the NLS equation has high mathematical complexity to study, especially in investigating an optical soliton (Lin, 2006; Syafwan, 2012). In a simple case, the solutions (solitons) of the NLS equation has been found through an analytical method, namely the inverse scattering method (Agrawal, 2013). In a more complicated case, finding analytical solutions (solitons) of the NLS equation is still a challenging problem. Therefore, many kinds of research are focus on the development of a numerical method as an alternative approach in solving the equation.

There are two main classifications of the numerical methods in solving the NLS equation, namely finite difference and pseudospectral methods (Agrawal, 2013). Taylor (2017) has shown a comparison of the two methods in solving the NLS equation and obtained that the pseudospectral is more straightforward and more accurate. In this case, the type of pseudospectral method used is the numerical split-step Fourier method, which is known to be relatively faster and more powerful compared to the finite difference method. This fact is because most of the process in the split-step Fourier method is performed merely through the fast Fourier transform (FFT) algorithm (Zen et al., 2002; Taylor, 2017).

This paper is organized as follows. In Section 2, we construct mathematical and numerical formulations of the NLS equation. We solve the equations by the numerical split-step Fourier method. In this section, we also provide an algorithm to evaluate the numerical solutions of the equation. Next, in Section 3, we discuss the results of the analytical and numerical calculations of the NLS equation and present the comparison between these solutions as a benchmarking of the split-step Fourier method. In the last section, we make some conclusions.

2. METHOD

This research is carried out theoretically. Therefore, a systematic framework is needed to explain the occurred physical processes. The conceptual framework is the benchmarking of the split-step Fourier method in solving a soliton propagation equation in a nonlinear optical medium. In achieving the objective, a systematic research method is arranged as follows.

2.1 Mathematical Formulation

The mathematical model used in this study is the NLS equation which describes the propagation of solitons in an optical fiber. In this case, the NLS equation reads as

$$i\frac{\partial U}{\partial \xi} = \operatorname{sgn}\left(\beta_2\right)\frac{1}{2}\frac{\partial^2 U}{\partial \tau^2} - N^2\left|U\right|^2 U. \tag{1}$$

In Eq. (1), $U = A/\sqrt{P_0}$ is an envelope function that emitted into optical fibers, with A and P_0 respectively describe the envelope amplitude and peak power. Next, $\xi = z/L_D$ and $\tau = T/T_0$ represent space (spatial) and time of propagation in dimensionless form, L_D and T_0, respectively indicate the length of the dispersion area and the width of the envelope. Furthermore, $\operatorname{sgn}\left(\beta_2\right) = \pm 1$ states the envelope area in the optical fiber; +1 corresponds to normal fields while -1 correspond to anomalous regions experiencing dispersion disorders. Finally, N is a nonlinear parameter of an optical fiber (Agrawal, 2013).

Agrawal (2013), Eq. (1) is derived from Maxwell's equation. It is known to have an analytical solution obtained by the so-called inverse scattering method. By this method, the analytical solution of Eq. (1) is given by

$$U(\xi,\tau)=\operatorname{sech}(\tau)\exp\left(\frac{i\xi}{2}\right). \tag{2}$$

Solution (2) will be compared with the corresponding solution obtained by numerical calculations in the next section.

2.2 Numerical Formulation

In this section, we construct numerical formulation to solve Eq. (1) by applying the split-step Fourier method. Following Zen et al. (2002), Agrawal (2013) and Taylor (2017), rewrite Eq. (1) in the form

$$\frac{\partial U}{\partial \xi}=\left(\hat{D}+\hat{N}\right)U, \tag{3}$$

where

$$\hat{D}=-i\frac{\operatorname{sgn}(\beta_2)}{2}\frac{\partial^2}{\partial \tau^2} \tag{4}$$

and

$$\hat{N}=iN^2|U|^2. \tag{5}$$

Here, $\hat{D}$ is a differential operator that takes into account the dispersion properties of an optical fiber, while $\hat{N}$ is a nonlinear operator that regulates the effect of optical fiber nonlinearity.

The rationale for the split-step Fourier method is based on the analysis of structural problem (Zen et al., 2002; Agrawal, 2013). In this case, the envelope, which is subject to the dispersion and nonlinearity of an optical fiber, is segmented. The form of segmentation is illustrated in Figure 1.

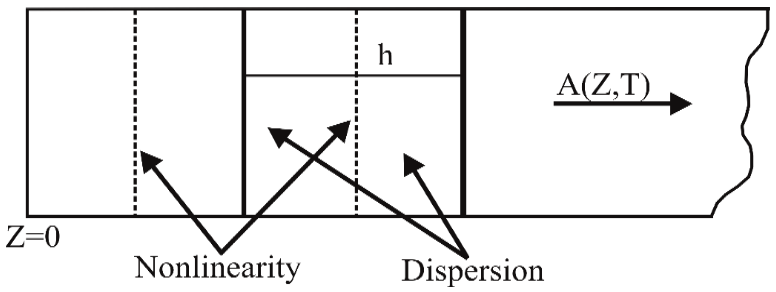

Figure 1 Illustration of the split-step Fourier method: effect of symmetry of dispersion and nonlinearity along optical fiber (Zen et al., 2002)

From Figure 1, it is shown that the dispersion and nonlinear properties in optical fibers act together. Therefore, the solution of Eq. (3) by using the split-step Fourier method can be determined by assuming that when the envelope propagates at a small distance (h), the dispersion and nonlinear effects act independently. In a further review, the envelope propagation problem from z to $z+h$ can be solved in two steps. In the first step, nonlinearity acts alone ($\hat{D}=0$), while in the second step, dispersion acts alone ($\hat{N}=0$) (Agrawal, 2013; Taylor, 2017).

Based on the two steps, we can approximate the solution of Eq. (3) in the form

$$U(\xi+h,\tau)\approx \exp\left(h\hat{D}\right)\exp\left(h\hat{N}\right)U(\xi,\tau). \tag{6}$$

This solution can be simplified using the Baker-Hausdorff formula (Zen et al., 2002), i.e

$$\exp(\hat{a})\exp\left(\hat{b}\right)=\exp\left(\hat{a}+\hat{b}+\frac{1}{2}\left[\hat{a},\hat{b}\right]+\frac{1}{2}\left[\hat{a}-\hat{b},\left[\hat{a},\hat{b}\right]\right]+\cdots\right), \tag{7}$$

by assuming it is independent of z (Agrawal, 2013). In Eq. (7), $[\hat{a},\hat{b}]=\hat{a}\hat{b}-\hat{b}\hat{a}$ takes the form of a commutator. So by definition, $\hat{a}=h\hat{D}$, $\hat{b}=h\hat{N}$ and ignoring the non-commute term (since it has an uncertain value), the solution (6) can be written in a more straightforward form as

$$U(\xi+h,\tau)=\exp\left\{h\left(\hat{D}+\hat{N}\right)\right\}U(\xi,\tau). \tag{8}$$

In principle, the solution (8) contains a time differential component ($\partial/\partial\tau$), as shown in Eq. (4). Therefore, it becomes complicated in the algorithmic process. Thus, the exponential operator containing in (8) must first be evaluated in the frequency domain (ω) using the Fourier transform given by Suarez (2013) as follows

$$\tilde{\hat{O}}\left(\frac{\partial}{\partial\tau}\right)=f\left(\hat{O}(-i\omega)\right), \tag{9}$$

$$\hat{O}(-i\omega)=f^{-1}\left(\tilde{\hat{O}}\left(\frac{\partial}{\partial\tau}\right)\right) \tag{10}$$

From here, we can express Eq. (8) in the form

$$U(\xi+h,\tau)=f^{-1}\left[\exp\left\{h\hat{D}(-i\omega)\right\}\cdot f\left[\exp\left\{h\hat{N}\right\}\right]U(\xi,\tau)\right], \tag{11}$$

where f and f^{-1} are the Fourier transform and its inverse. The purpose of the opposite in this transformation is to return the exponential operator to the time domain.

Eq. (11) is a numerical solution of the NLS equation (1) in optical fibers using the split-step Fourier method. Through this solution, we will evaluate how the soliton profile generated from Eq. (1). We will also examine the accuracy of the split-step Fourier method as a benchmarking in solving the soliton propagation equation (1). The solution (11) is known to be easily or quickly evaluated using the fast Fourier transform (FFT) algorithm.

2.3 Algorithm

In this section, we evaluate Eq. (11) to see the soliton profile in optical fibers. This profile can be obtained through the following algorithm (Zen et al., 2002; Agrawal, 2013):

1. Determine the optical fiber input parameters, i.e. optical fiber length (L_D), envelope group-velocity dispersion (GVD) parameter (β_2), nonlinear parameter (N), and initial condition of the envelope $U(0,\tau)$.
2. Determine the magnitude of change in z ($h=\Delta z$), the total number of z iterations (*step_num*), number of grids in the time domain (nt), and time interval (Δt) given by $2T_{\max}/nt$.
3. Perform the calculation of Eq. (11) as many times as the *step_num* iteration by taking the operators (4) and (5).
4. Plot the numerical solution U against coordinates τ .

We will compare the numerical solution obtained through the above algorithm with the analytical solution (2). The comparisons are presented in visual forms to see the benchmarking of the split-step Fourier method.

3. RESULT AND DISCUSSION

The main discussion in this paper is the benchmarking of the numerical split-step Fourier method in solving the NLS equation (1). Let us first present the visualization of analytical and numerical solutions of the equation.

3.1 Analytical Solutions

Analytical solutions of the NLS equation (1) can be calculated from the inverse scattering method, which is given in Eq. (2). The visual form of the analytical solution is shown in Figure 2.

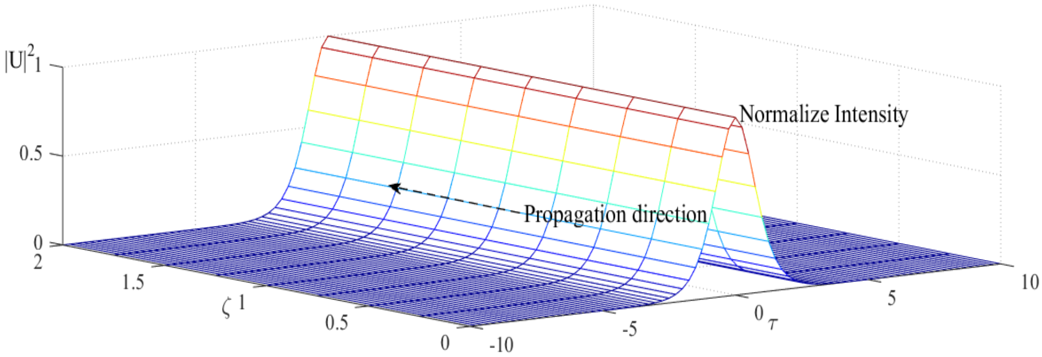

Figure 2 Soliton profile from the analytical solution of the NLS equation (1)

The solution profile in Figure 2 represents the form of soliton in an optical fiber, namely in the type of a temporal soliton. Temporal soliton is the first type of soliton found in an optical fiber (Kisvhar and Agrawal, 2003; Kajzar and Reinisch, 2006; Agrawal, 2013). This type of soliton is known to occur when a beam applied in an optical fiber is limited in spatial. Besides, this type of soliton also has a unique envelope shape manifested in the time domain, as shown in Figure 2.

Physically, solitons in this system occur when the nonlinear effect causes a self-phase modulation (SPM) in an optical fiber to balance the dispersion effect that responds to change in soliton velocity from the envelope group velocity when it propagates along the optical fiber (see Agrawal, 2013). SPM is known to produce changes in the refractive index of optical fiber, which causes the envelope of the beam to be disturbed (narrowed) along with the optical fiber. The dispersion effect of the optical fiber is known to create envelope velocity dependent on frequency and causes the envelope to widen. When the two effects balance each other, stability is emerged and then form the so-called soliton (see again Figure 2).

3.2 Numerical Solutions

The numerical solution of the NLS equation (1) using the split-step Fourier method is calculated from Eq. (11). We will evaluate the solution according to the algorithm presented in Section 2.3. For input parameters, we consider three cases, i.e. (a) $L_D = 10,\ \beta_2 = 1,\ N = 1$, (b) $L_D = 10,\ \beta_2 = -1,\ N = 1$, and (c) $L_D = 10,\ \beta_2 = -1,\ N = 2$. These parameters become important to look at the soliton characteristics resulting from the NLS equation (1). The visual form of the numerical solution is given in Figure 3.

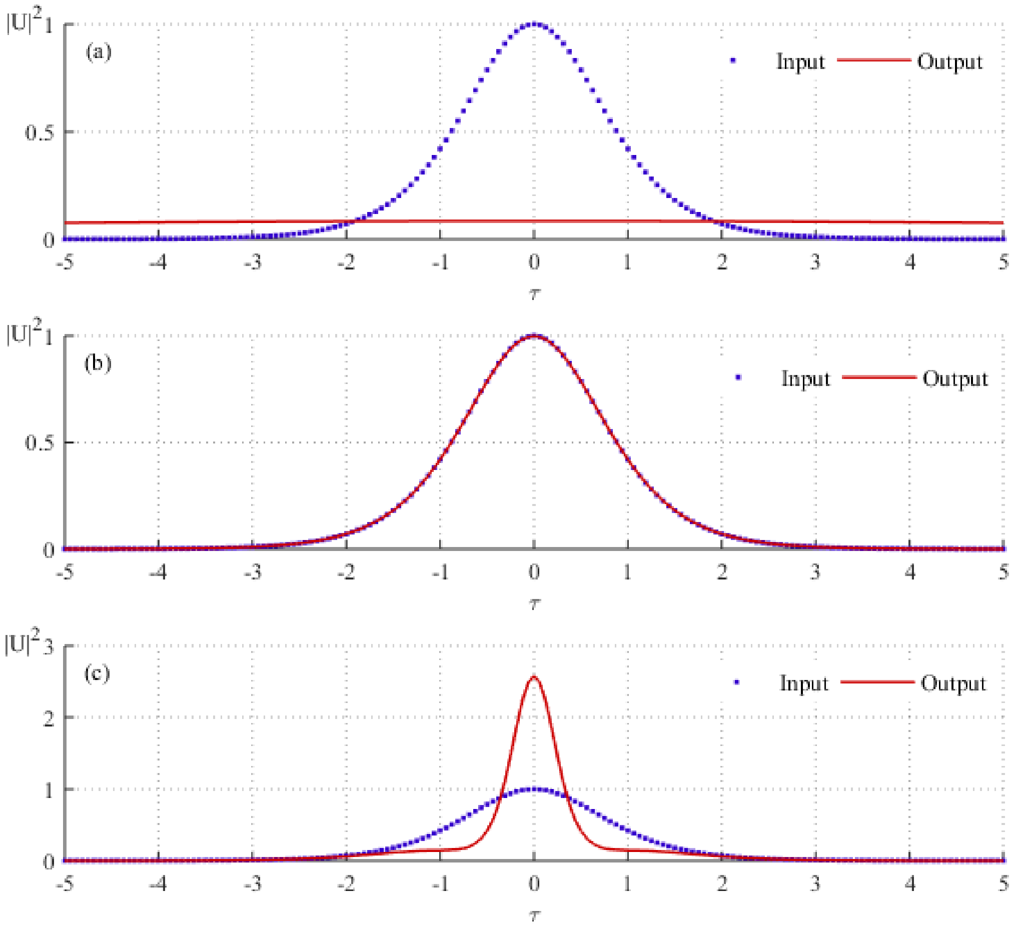

Figure 3 Numerical solution of the NLS equation (1) with input parameter values: (a) $L_D = 10,\ \beta_2 = 1,\ N = 1$, (b) $L_D = 10,\ \beta_2 = -1,\ N = 1$, and (c) $L_D = 10,\ \beta_2 = -1,\ N = 2$

Figure 3 shows the envelope profiles of the numerical solutions of the NLS equation (1). These profiles are obtained from the secant-hyperbolic input pulse ($\operatorname{sech} \tau$). The blue profiles in Figure 3 show the input pulses, and the red ones indicate the corresponding output pulses. The output pulses are in the form of temporal solitons, which are, in principle, consistent with the analytical solutions discussed in Section 3.1. In Figure 3, it is also shown that when $\beta_2 = 1$ (i.e. in the normal GVD region), a soliton does not exist. Otherwise, when $\beta_2 = -1$ (i.e. in the anomalous GVD region), a soliton exists. This fact confirms the existence of soliton in the anomalous GVD region, as also reported in several references, one of which is Agrawal (2013).

In Figure 3, it is also confirmed that the stability of solitons is affected by the nonlinear nature of optical fibers (i.e. parameter N). We can see in Figures 3(b) that for $N = 1$ solitons are stable, indicated by the amplitude and width of the input pulse are equal to the output. Otherwise, in Figure 3(c), for $N = 2$ solitons experience instability, i.e. the amplitude of output pulse is higher, and the width is narrower. This instability occurs as a consequence of envelopes or pulses that propagate in a nonlinear medium (narrowed) (Agrawal, 2013).

3.3 Accuracy of the Split-Step Fourier Method

In this section, we present a visualization of the accuracy of the split-step Fourier method in solving the NLS equation (1). Previously, we have provided the analytical solution of the equation in Section 3.1, as presented in Figure 2, and the corresponding numerical solution at the beginning of Section 3.2, as shown in Figure 3. The comparison of the two solutions is given in Figure 4.

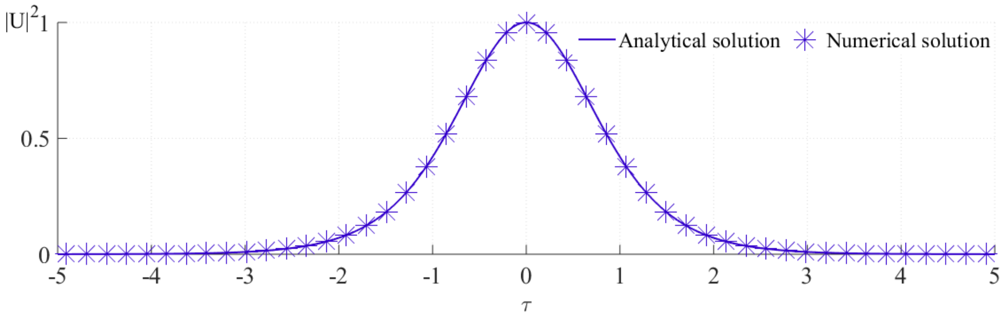

Figure 4 The numerical vs analytical solutions of the NLS equation (1)

Figure 4 shows that the analytical and numerical solutions are consistent. This fact suggests that the split-step Fourier method is powerful to apply in solving the NLS equation in a nonlinear optical medium.

4. CONCLUSION

Based on the results of numerical calculations, we conclude that the numerical split-step Fourier method is accurate to apply in solving a soliton propagation equation in a nonlinear optical medium. This method provides a relatively simple process and powerful for the NLS equation case. We confirmed the existence of solitons in the NLS equation propagating in a nonlinear optical medium.

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